Exploring the ways that research, theory, and working hypotheses, can broaden conceptions of mathematics knowing and doing.

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Invited Paper.

Abstract.

What competencies are brought to bear when students work on mathematics problems? And to what extent may these be represented by knowledge? These are questions that I consider in this article, as I explore notions of competency that go beyond knowledge to include broader aspects of mathematical 'proficiency' including the mathematical 'practices' in which people engage. This exploration will draw from two frameworks that have recently been introduced in the US. In addition, I consider the ways in which research knowledge is conceived and developed, reflecting upon the important role of theory and the potential of 'working hypotheses' for connecting with practice in new ways.

Introduction

It is some years now since the day I received an important envelope in my mailbox in London, carrying the ESM postmark. I opened it nervously, knowing that the contents would tell me whether my first ever article had been accepted. To my delight it was and that time marked not only my own beginning relationship with the journal, but with the scholarship of mathematics education. I was a masters student at the time, and unsure whether I could contribute to the world of academia. I had submitted my paper to ESM, on the advice of one of my university professors, who told me it was one of the most important journals in mathematics education. Publishing in ESM was a wonderful way to begin my writing career, as the editor at that time, Leen Streefland, was supportive and encouraging in helping me prepare my article for publication. Since that time ESM has continued to play an important role in my work, providing a forum through which I learn from research and ideas produced all around the world, indeed its international scope is one of the aspects I most appreciate about the journal. It was with great pleasure then that I received an invitation to write this paper, as part of the celebration of the 50th anniversary of the journal. I was asked to do so, as a current user of the journal, someone who publishes in ESM and uses it in teaching and research. I have therefore chosen to reflect upon some ideas that I am working on now, that ESM has helped stimulate, problematise and nourish, through articles that have appeared. I will consider in this short paper, what it means to have a broad conception of knowing - for research and for mathematics - reflecting upon the contribution of a selection of ESM articles as I do so.

Using Theory to Advance Knowledge.

The task of researchers and scholars in mathematics education, as in other academic fields, is to create new knowledge. This is no small endeavour and it carries with it a critical responsibility to be open to new ideas, perspectives and ways of thinking. For if scholars do not entertain *other* ways to think, then scholarship is in danger of closing in, and of moving only in concentric, or ever decreasing circles. Theory is critical to the production of research knowledge, and to educational work more generally. Stephen Ball warns of the danger of educational studies becoming removed from theory (as encouraged by some government initiatives, such as removing teacher education from universities), arguing that the abandonment of theory will change teaching and education from an intellectual to a technical process. He contends that the value of theory is that it:

> 'can separate us from 'the contingency that has made us what we are' (…) Theory is a vehicle for 'thinking otherwise' (…) It offers a language for challenge, and modes of thought, other than those articulated for us by dominant others. It provides a language of rigour and irony rather than contingency. The purpose of such theory is to de-familiarise present practices and categories, to make them seem less self evident and necessary, and to open up spaces for the invention of new forms of experience' (1995, p266).

One of the roles of theory, as Ball notes, is to protect us from the limits of our own experience, which inevitably narrow our thinking. Theoretical frameworks encourage researchers to pursue new ideas, prompting selections and explorations of data that our experiences would not initiate. Theory also serves as an interpretive tool, helping researchers to understand particular interactions that take place, by positioning them - providing dimensions along which ideas may be located and examined. McDermott and Lave (2002), like Ball, urge the careful use of theory as a way of prevailing against ideological subjection and dogma:

> 'We cannot trust ourselves to think our way to ideas that we need to change our lives. We need help. One kind of help is to work on rich texts that force us systematically to relocate our work with the work of others' (McDermott & Lave, 2002, p46).

But researchers in mathematics education do not only need to use theory, they need to select theoretical frameworks carefully. McDermott and Lave allude to the importance of theory in tackling big issues of social justice and equality, in producing work that may 'change lives' (p46). They argue that narrow frameworks may not give us the perspective to question prevailing practices with which we are familiar and which it is hard to see beyond. In 1990 Yves Chevallard took up a similar issue in ESM, criticizing research in mathematics education for its narrowness. He claimed that:

> 'research in mathematics education must certainly broaden its outlook, and take into account determinants which it has so far flippantly ignored. It is also its duty, nevertheless to investigate patiently, even punctiliously, the relationship between the individuals' experience and conduct and the socially determined contexts in which they emerge.' (Chevallard, 1990, p. 24).

Chevallard spoke to the importance of broadening the questions mathematics educators had been asking, the variables considered, and the frameworks employed, as well as drawing connections between different frameworks. Whereas Ball warned against the dangers of becoming atheoretical, Chevallard's article reminded us that the singular adoption of particular theories or frameworks can also be narrowing.

The vast majority of early inquiries in mathematics education drew from the same, or similar perspectives and frameworks, and many would acknowledge that this helped our field to gain strength and to develop a clear identity. It also allowed knowledge to cumulate in careful ways. Researchers who studied students' mathematical conceptions, for example, worked collectively across a number of countries to map out progressions of subject understanding in a few key areas (for example Hart, 1981; Kuchemann, etc). But the field has now diversified (Lerman, 2000), and researchers draw from many different methodologies and frameworks. This has prompted some crises in identity (Sierpinska & Kilpatrick, 1998), and increased demands for new researchers entering our field who need to learn about previous work, but it also paves the way for many new and important inquiries in the future of mathematics education.

This point was well illustrated in an important article that appeared in ESM in the mid nineteen-nineties. At a time when constructivism had become a dominant paradigm in mathematics education, displacing previous theories of learning, Robyn Zevenbergen (1996) wrote an article in which she questioned the limits of constructivism, claiming that it had attained an unhealthy dominance within the field. She was particularly concerned about its inability to 'theorise adequately the marginalisation of significant numbers of students' (1996, p96). In this important essay Zevenbergen illustrated the role of theoretical breadth, of the use of multiple theories to counter dominant hegemonies. One of Zevenbergen's main propositions was that success in mathematics is not simply a matter of cognitive processes, and that 'students from certain social and cultural groups are more likely to be constructed as effective learners of mathematics because of their congruency with the social context of formal schooling' (p. 105). Zevenbergen drew from sociological theory to argue this powerful idea, encouraging researchers to locate understandings of students' engagement and success within a broad sphere, that extended beyond students' interactions with curricular materials. This idea emerged through the careful juxtaposition of different theories, including constructivism and cultural capital (Bourdieu, 1983, 1985) as Zevenbergen explored their differences and the insights they gave.

Ball, Chevallard, Lave, McDermott, and Zevenbergen all concern themselves with an extremely important issue – that of breadth of thought and of open-ness. They remind us that theoretical perspectives play an important role, prevailing against ideology and dogma. They also remind us that theories must be employed with care and reflectivity, in order to preclude a form of narrowness that comes from the unquestioning acceptance of dominant paradigms. Chevallard and Zevenbergen urge mathematics educators to look beyond single frameworks, remaining open to the different ways that theories from within and outside mathematics education may illuminate some pressing and enduring questions in our field, such as those of social inequality. The different authors speak to the importance of a research knowledge that is broadly conceived, enhanced by theoretical reflexivity.

But what does it mean to be aware of diverse epistemological perspectives, to show appreciation for them and to draw upon them? If one surveys the landscape of mathematics education research it is possible to find many examples of scholars who have embraced different epistemological positions and produced important new knowledge because of such openness. In the past, mathematics education researchers drew largely from psychological frameworks and theories, but contemporary researchers are increasingly demonstrating the insights that may be learned from additional frameworks. Central among the frameworks now drawn upon are sociology (Bauersfeld, 1995; Ensor, 2001; Dowling, 1996, 1998; Restivo, 1992; Morgan, 2000); sociocultural theory (Kieran, Forman & Sfard, 2001; Lerman, 2001); politics (Skovsmose, 1994; Valero, 1999; Vithal & Skovsmose, 1997); mathematics (Ball & Bass, 2000); philosophy (Ernest, 1991, 1999) history (Gerdes; Joseph, 1992) and anthropology (Chevallard 1992; Artigue, 1999). In these different accounts the researchers draw from varied disciplinary perspectives and make use of methodological frameworks that are associated with them. But scholarship is anything but simple, and whilst we may agree that breadth of thinking is critical to the evolution of ideas, and that different frameworks should be considered and employed, we must also be wary that mathematics education is a relatively new and young field (Kilpatrick, 2002) and that too much breadth and diversity will cause a scattering of focus and preclude opportunities for consolidation and identity (Sierpinska & Kilpatrick, 1998). Indeed work that starts from the findings of previous research, and builds from such findings, to provide depth and texture, such as the history of work on misconceptions and cognitive change contributes significantly to our understandings. Over time we must hope that journals continue to support both types of work, as our field diversifies and grows. Indeed research journals, such as ESM, play an extremely important role in providing an avenue of communication for different theories, as well as arguments about the use of theory. Over the years ESM has continued to encourage breadth and open-ness of thought through the publication of articles such as Zevenbergen's that challenge dominant ways of thinking, as well as through its encouragement of different genres of article; different modes of research; and different frameworks of analysis.

Relations of Knowledge and Practice

The ways in which theory may support work in the scholarship of mathematics education is an important question for our field, but an equally important question to consider is the ways in which theory may impact the practices of mathematics teaching and learning. Educators, unlike some other scholars, are working in an applied field and our work is judged, in part, by the extent to which it is able to impact – indeed improve – educational practice. It is in this area that scholarship in education is most often found to be lacking. Greeno, McDermott, Cole, Engle, Goldman, Knudsen, Lauman and Linde (1999) provided an interesting perspective on this issue. Greeno and his colleagues suggested a new conceptualisation for knowledge and theory in education. They provocatively claimed that we remove the boundaries between knowledge and 'domains of practical activity' (1999, p303) suggesting that a means of doing so would be to engage teachers, researchers and students in new participation structures, in which they worked together to produce new research knowledge. Such structures are not, themselves provocative, but Greeno *et al* propose that 'expert knowledge is better seen as a working hypothesis that must enter a community of practice and jostle apparent knowledge until it takes root in a reorganization of what people can do with each other' (1999, 301-302). Drawing from both Dewey and Mead, they offer a different conceptualisation of knowledge as a 'set of working hypotheses that would be tested against their consequences for the body politic' (p333). Thus some of the knowledge produced by educational scholars should be judged against the extent to which it impacts practice. This view is echoed in a National Academy of Science report (NAS, 1999) from the US that suggests that educational research, in future, should be judged not only on its appeal to other academics, or its *potential* to improve education, but on its impact. The report proposes a change for educational funding, with priority given to those studies that aim to impact practice, and that bring together researchers and practitioners who would work together to frame problems of practice, and their solutions.

The NAS group recommends that educational research is reorganized with new kinds of support, review, participation and evaluation, all organized to enable the collaboration of different 'communities' such as researchers, practitioners and developers of materials. They support a kind of research that they call 'problem solving research and development' - characterised by researchers and practitioners identifying problems of practice together and working together to solve them. In doing so they challenge an assumption that research knowledge should be developed as general principles that should then be conveyed to teachers and other education professionals. They replace this vision of research and dissemination with one in which the development of knowledge, understanding and educational improvement takes place as one process, shared by specialists in research, development and educational practice.

The proposal to re-conceptualise knowledge as a set of working hypotheses, tested against their ability to impact practice, is an interesting one for a journal such as ESM. Greeno *et al* do not suggest that such knowledge should replace discipline based knowledge that is valued for its own sake, but they argue for an expansion of the ways research knowledge is developed and valued in our field. I began this article with discussion of the ways knowledge may be enriched by theory - this is another call for the enrichment of knowledge, this time by practice, and it raises an interesting question for journals such as ESM. Perhaps now, on the anniversary of its 50^{th} birthday, at a juncture where there are serious dichotomies between research knowledge and the practices of education, it is time for ESM to give its authors new directions that could change the way research knowledge is developed and used. These could include instructions for authors to reflect upon the ways their research findings may be used in practice, and the means by which they will be communicated to, and taken up by practitioners to improve students' educational opportunities. This does not mean that new knowledge would not be valued for its own sake, or that there would be no place for conceptual essays in ESM. But perhaps researchers could be urged to consider their findings as 'working hypotheses' and explore the ways in which their knowledge may enter different communities and 'jostle apparent knowledge until it takes root in a reorganization of what people can do with each other' (Greeno *et al*, 1999, 301-302)? Whether this is feasible, desirable, or possible, is a question that seems worthy of consideration.

The expansion of research knowledge – to benefit fully from theory, and to connect in new ways with practice, is an issue that has been furthered by a few important contributions in ESM, that call for a prevailing open-ness in our conceptions of knowledge. But it is knowledge of mathematics itself, rather than research knowledge, that has gained the greatest attention of ESM authors. Indeed the question of what it means to know mathematics, and be proficient in its use, is one to which researchers of mathematics education have contributed a great deal. I will end this paper with some consideration of the domain of mathematics itself, as explorations of the ways mathematics knowledge may be opened, expanded, and enriched have an important role to play in our work as researchers of mathematics education, as I shall argue below.

Expanding Conceptions of Mathematical Knowing and Doing.

Many articles over the years have reported on research that has considered how mathematics is learned – which approaches are effective, for whom and why. This work is at the centre of our field and is extremely important. A related question, that has also received considerable attention, concerns the nature of mathematics knowledge – what is it, and how is it, or could it, be held?

Ball, McDermott, Lave, and others all stress the importance of thinking openly about educational issues, using theory to protect against narrow interpretations constrained by personal experience. The issue of open-ness is also critically important for those who consider the nature of mathematics knowledge, particularly because the prevailing dogma about what it means to know and be proficient in mathematics is extremely narrow in most countries (Boaler, 2002a). Indeed, one could argue that it is the narrowness with which mathematics is regarded that has maintained a system of educational failure, in which only a few ever attain mathematical proficiency or fluency. In the two countries in which I have worked – the UK and the US – mathematics is believed by many to be a collection of disconnected, standard procedures. In schools, homes, and departments of education, test success is held as the ultimate goal, and many students of mathematics believe that their goal is to memorise numerous different, unrelated procedures, so that they can reproduce them when they are given different test questions. This scenario can lead to a limited test-knowledge, or worse, as students often fail even to be successful on tests, as they find that memorised procedures are insufficient when faced with questions that require that they also know *when* different procedures should be selected (Brown, 1981; Boaler, 1997, 2002b). Research in mathematics education has contributed a great deal to our understanding of these problems, and various authors in ESM have been particularly helpful in exploring the boundaries of mathematics knowing.

In the nineteen-seventies and eighties there was important work, and agreement about different forms of knowledge that may be developed, that have been variously characterised as conceptual and procedural (Hiebert, 1986), or relational and instrumental (Skemp, 1976). These characterisations of knowledge have been important, and they have resonated with many teachers and scholars. But recent articles in ESM capture some different, important aspects of mathematical work that seem to elude a knowledge characterisation. Indeed when we step outside of knowledge domains, it seems that our field may lack agreed frameworks for understanding mathematical activity. As an example consider an excellent article by Noss, Healy, and Hoyles (1997). In this research paper they describe the ways in which they developed a particular 'microworld', which was designed to 'help students construct mathematical meanings by forging links between the rhythms of their actions and the corresponding symbolic representations they developed' (p203). In this article Noss, Healy, and Hoyles examine the ways in which students *make connections* between visual and symbolic forms of functional relationships. They argue, importantly, that students often develop a disconnected sense of algebraic formulations – regarding algebra as an end-point, rather than a problem-solving tool. The authors therefore designed a computer environment in which the only way to manipulate and reconstruct objects was to explicitly express the relationships between them. In doing so they helped the students view and use algebra as a representational tool, in the service of the expression of mathematical connections and relationships.

Noss, Healy, and Hoyles (1997) argue that the act of making connections is important because mathematical meanings *derive* from mathematical connections. Thus they link mathematical connections to the knowledge and understanding they may promote. But it seems to me that the act of observing relationships and drawing connections, whether between different functional representations or mathematical areas, is a key aspect of mathematical work, in itself, and should not only be thought of as a route to other knowledge. In Leone Burton's article of 1999, she reported upon a study in which she interviewed 70 research mathematicians, probing the ways that they viewed and used mathematics. One of the key aspects of their work that the mathematicians highlighted was also the act of making connections - as one of them reflected:

> ‘The behaviour that I observed I couldn’t find anything about it in the standard literature. Then I found a connection with a very ancient problem in gravitational mechanics and I found some old computational work from the 1960’s and the behaviour they found was almost identical but even richer than the behaviour I was finding. So whilst I was understanding more and more about my problem, I was also seeing how it was linking into this huge area of the 3-body problem … That was really nice. I think that is pretty common in maths. Things do connect when you don’t expect them to. (Male lecturer).’ (Burton, 1999, p136).

Andrew Wiles made one of the most significant mathematical breakthroughs of our time in his proof of Fermat's Last Theorem. Biographical accounts of his work (eg Singh, 1998) highlight the significance of the connections Wiles was able to draw between different mathematical theories; indeed it was the connections that he and other mathematicians, were able to draw that laid the path for the eventual proof. Frey, a German mathematician, set the ball rolling when he claimed that 'if anyone could prove the Taniyama-Shimura conjecture then they would also immediately prove Fermat's Last Theorem'. Ken Ribet and other mathematician's worked hard to complete the connection between the Taniyama-Shimura conjecture and Fermat's Last Theorem, but could only prove 'a very minor part of it' (p201). But, as Singh recalls:

> 'Fermat's Last Theorem was inextricably linked to the Taniyama-Shimura conjecture. (…) For three and a half centuries Fermat's Last Theorem had been an isolated problem, a curious and impossible riddle on the edge of mathematics. Now Ken Ribet, inspired by Gerhard Frey, had brought it center stage. The most important problem from the seventeenth century was coupled to the most significant problem of the twentieth century. A puzzle of enormous historical and emotional importance was linked to a conjecture that could revolutionize modern mathematics.' (Singh, p202).

Wiles subsequently went on to prove the Taniyama-Shimura conjecture, thereby proving Fermat's Last Theorem, a breakthrough that derived from the connections drawn.

Burton and Singh both write about the work of mathematicians, focusing upon the act of ‘making connections’. But what is this aspect of mathematical work? And has its character - as an action or mathematical practice - rather than a form of knowledge or knowing - contributed to its relative lack of attention in curriculum materials and teaching? For the act of making connections is not something students need to *know*, it is something they need to *do*. One could imagine a student with a broad knowledge of mathematical procedures and even a conceptual understanding of the relations between procedures, who still would not think to draw connections between different mathematical ideas, relations or representations as they work. They may do so, as knowledge and practice are intricately connected, but the act of doing so is not defined by the knowledge they possess. This raises the question of whether mathematics education researchers have focused too predominantly upon knowledge categories, neglecting various mathematical actions, such as those to which Burton; Noss, Healy, and Hoyles draw our attention, that are so critical to mathematics work. Before pursuing this question further I will consider another example of research, published in ESM, that highlights a second aspect of mathematical work that eludes knowledge categorisation.

In 1996 Marty Simon drew our attention to what he calls a mathematical ‘ability’ - that of considering a mathematical problem, with the various constraints and variables described, not as a static state, but a dynamic process. Simon gives as an example a girl who is considering whether she can make an isosceles triangle, using geometry software, if only two angles and the included side are specified. The girl immediately represents such a triangle and justifies it saying ‘Well, I know that if two people walked from the ends of this side at equal angles towards each other, when they meet, they would have walked the same distance’ (1996, p199). Simon argues that such reasoning is not inductive – the girl did not generate several triangles and notice a pattern. Nor is the reasoning deductive – she did not make a conjecture and create a need for a deductive proof. Rather she saw the isosceles triangle not as a static figure with particular dimensions, but a ‘dynamic process that generates triangles from the two ends of a line segment’. This dynamic mental model, as Simon points out, enabled her to reason about two ideas, that she connected – the relationship between the base angles of a triangle, and the relative length of the legs of the triangle given particular angles. Simon describes this mode of work as 'seeking a sense of how the mathematical system works' or developing a 'feel for the system' (p198) and refers to this process as 'transformational reasoning' (1996, p197).

Simon, like Burton, Noss, Healy, and Hoyles, describes a particular action – of viewing mathematical relationships as a dynamic state. This action, like that of noting relationships or drawing connections, also eludes knowledge characterisation, and Simon refers to it as a particular type of reasoning. Ben-Zvi and Arcavi (2001) analyze the work of students who were learning to use exploratory data analysis in a technological environment. In their descriptions of the students working Ben-Zvi and Arcavi highlight the importance of what they call 'habits' such as questioning, representing, concluding, and communicating. Ben-Zvi and Arcavi stress the importance of enculturation as an act of teaching, with teachers inducting students into statistical work, so that they may learn a variety of 'thinking processes' and 'problem solving strategies'. Their focus in this paper also extends beyond the knowledge that students need to explore data, to some important mathematical actions, such as 'looking globally at a graph as a way to discern patterns and generalities' (2001, p38). The different authors I have reviewed all describe critical mathematical actions that extend beyond knowledge - but when we consider how such actions are characterised or defined, things look a little hazy. Some authors have broadened notions of knowledge and argued that many actions may be considered as different *forms* of knowledge. In a special edition in 1999 (Tirosh, 1999) for example, a range of authors described different categories of knowledge such as implicit, explicit, formal and visual knowledge, as well as knowing that, knowing why and knowing how. These notions extend traditional conceptions of knowledge, enabling researchers to account for broader aspects of competent performance, but it seems that the different actions people employ are not <u>defined</u> by this knowledge. Other scholars have described the kind of actions I have reviewed as mathematical *processes*, with names such as reasoning, and communicating (Schoenfeld, 1985, 1992; NCTM, 2000). At this time there is little agreement about the nature, or form of mathematical actions, such as visualising, or connecting, and this may be part of the reason that these critical aspects of mathematics work are frequently overlooked in curriculum planning and assessment.

In these different characterisations of the mathematical work employed by students and mathematicians we also gain an important sense of some mathematical traits that supported the work, including creativity, interest, and inquisitiveness. Indeed it is hard to believe that such characteristics could be separated from the work of drawing connections or regarding relations dynamically. Yet these characters are similarly elusive and difficult to define - sometimes being considered as mathematical 'beliefs', at other times 'habits of mind' (Cuoco, xx). Such habits, or beliefs are also given little attention when policy makers and schools make decisions about curriculum materials, and teaching strategies. This lack of attention, I would argue, has been encouraged by the separate study, and labelling, of characteristics such as mathematical knowledge, 'processes' and 'habits of mind'. Thus scholars and teachers have often focused upon knowledge as though it develops independently of belief, action, or disposition. This is particularly true of studies of students' knowledge - scholars of teacher knowledge have focused to a greater extent upon the relationships between teacher knowledge and belief (see, for example, Even & Tirosh, 1995; Cooney, 1991; Thompson, 1992) whereas it is still relatively commonplace for studies of student learning to focus only upon knowledge developed. In my own recent work, analyzing students' learning of mathematics in different teaching approaches (Boaler, 1997, 2002a, b) I have found it useful to think of students' *mathematical identity* - their relationship with the discipline of mathematics (Boaler, 2002a, b). This idea of a disciplinary relationship seems helpful because it includes the knowledge a student possesses, but it also pays attention to the ways in which students hold knowledge, the ways in which they use knowledge and the accompanying mathematical beliefs and work practices that interact with their knowing. Situated theory has contributed a great deal to our understandings of the practices of classrooms that intersect with students' knowing, but such theories have centred upon the practices of systems and environments, considering the ways that these shape knowledge - with less attention to the practices of individuals, that may be thought of as an aspect of individual proficiency.

Two groups in the United States, comprising mathematics educators, teachers, mathematicians and policy makers, recently produced different conceptualisations of what it means to know and use mathematics that seem helpful in furthering our understanding of the different relations of mathematical knowledge, beliefs and actions. First the National Research Council (NRC) convened a group of experts, led by Jeremy Kilpatrick, to review and synthesize relevant research on mathematics learning, from pre-kindergarten to the end of grade eight (Kilpatrick, Swafford & Findell, 2001). As part of this work, the group provided a conceptualisation of successful mathematics learning, that they called 'mathematical proficiency'. The different, interwoven, aspects of proficiency they proposed, are:

> '*conceptual understanding* - comprehension of mathematical concepts, operations and relations
> *procedural fluency* - skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
> *strategic competence* - ability to formulate, represent and solve mathematical problems
> *adaptive reasoning* - capacity for logical thought, reflection, explanation, and justification
> *productive disposition* - habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.' (p116)

This representation of proficiency seems important, particularly in its inclusiveness, and its bringing together of knowledge, aspects of mathematical work, and disposition. Practices such as making connections, and viewing mathematical representations dynamically, could be included as examples of 'strategic competence', whereas characteristics such as creativity, and inquisitiveness may be included under 'productive disposition'. The utility of this framework will become clearer in time, as researchers, teachers and others work with the ideas, but it seems that it has the potential to do something very important - expand public conceptions of mathematics knowing and turn attention to the aspects of mathematical proficiency that need to accompany knowledge.

A second group of mathematicians and educators in the US, led by Deborah Loewenberg Ball, recently designed a proposal for future research directions in mathematics education. This proposal (reference?) is intended to guide government agencies in their distribution of research funds. The proposal has many interesting features, including a call for strategic accumulation of research in focused areas, and collaborations of different groups working on problems together. Most interesting for this discussion is the choice of one of the three proposed research directions for the future as that of 'mathematical practices'. The group recommends this area as a way of adding texture and understanding to the notion of mathematical proficiency, and they describe mathematical practices in the following way:

> 'This area focuses on the mathematical know-how, beyond content knowledge, that characterizes expertise in learning and using mathematics. The term "practices" refers to specific things that successful mathematics learners and users do. Justifying claims, using symbolic notation efficiently, and making generalizations are examples of mathematical practices.' (px).

The group, of which I was a part, was in agreement that the field would be significantly advanced by focused work on these practices - considering what they are, how they are learned, and how they are used, in the service of employment, recreation and mathematical inquiry. Examples of mathematical practices that the group highlighted for study, include justification, representation, and reconciliation. The actions of making connections (Noss, Healy, Hoyles, 1997; Burton, 1999), and seeing mathematical relations as a dynamic process (Simon, 1996) could also be taken as candidates for study in this area.

The notion of *practices*, central to situated theory (Cobb, Scribner etc), has been extremely generative in mathematics education, as researchers have begun to look beyond students' cognitive processes, to the norms of classrooms (Cobb etc) and the *learning practices* that are encouraged and that shape knowledge in different classroom systems (refs). The notion of *mathematical practices* attends in similar ways to the repeated actions in which people engage, but its main focus is not the learning of mathematics, but the *doing* of mathematics - the actions in which users of mathematics (as learners and problem solvers) engage.

Authors in ESM have contributed a great deal to our understanding of the subtleties of mathematical proficiency, and the various mathematical practices employed by learners of mathematics. Special editions of the journal on proof and reasoning (see for example, ESM, 1993, 14, 4 and ESM, 2000, 44, 1-2), and constructing meaning from data (ESM, 2001, 45 (1-3)), provide good examples of such work. The notion of practices, as well as that of mathematical proficiency, may provide useful frameworks for consideration of the different aspects of mathematical work that have been delineated. The journal's close attention to what it means to *do* mathematics, is in some senses not surprising, as Hans Freudenthal, one of the founding fathers of ESM, contributed a great deal to our understanding of mathematics as a problem solving act, a way of modeling and making sense of realistic situations (Van Oers, 2002). (See, for example, ESM 15, 1/2, for reflections on the 'legacy of Hans Freudenthal'). It is critical for our field that such work continues, and that we learn more about the nuances of mathematical proficiency that include and go beyond, knowledge. Such work certainly stands as an example of theoretical development that counters the dogmatism and ideology to which S. Ball, Lave and McDermott draw our attention. For if scholars do not consider the extent and nature of mathematical proficiency, in its broadest terms, we may be reduced to the dominant ideology that pervades public rhetoric, in which mathematical proficiency is equated with the reproduction of isolated mathematical procedures.

But whilst it is appropriate to commend the breadth of knowledge that has emanated from research, communicated in ESM and elsewhere, it is also sobering to reflect upon the gulf that exists between the understandings of mathematical proficiency communicated in ESM, and that which is communicated in many mathematics classrooms across the world. Greeno *et al*, as well as the NAS report from the US, provide some interesting proposals for ways of changing the situation and for building bridges between research and practice. These proposals include giving researchers the task of translating their findings into practical action and giving funders the task of prioritising research that takes place across academic and practitioner communities. I would add to those suggestions, a critical role for journals, such as ESM, in encouraging research that has a theory-practice dimension and giving potential authors new questions to consider regarding the links with, and impact upon practice, when submitting papers for publication.

Conclusion.

I was invited, in writing this paper, to reflect on the ways that ESM contributes to my work, as part of the celebration of the 50th anniversary of the journal. In exploring the ways that knowledge is produced and conceived - in the domains of research and mathematics - I hope to have acknowledged the critical contributions of ESM articles, in my own thinking and, more importantly, the development of our field. Articles such as those by Chevallard and Zevenbergen play an essential role in promoting theoretical awareness and reflexivity, inviting us to consider the frameworks we use and questions we raise in our research. Such articles have been extremely influential in my own work and in my teaching of future researchers, particularly masters and doctoral students.

In considering the ways our field conceives of mathematical knowledge, or proficiency, I have cited just a small selection of articles that have been instructive in expanding my understanding of mathematical work. The new knowledge produced in such work feeds my ongoing research work with students and teachers in schools and my teaching and professional development work with beginning and experienced teachers. It is interesting to consider how the world would be different, how proficient students may be, if teachers, parents and policy makers, paid greater attention to the different aspects of mathematical proficiency that researches have defined and that have been communicated in ESM. The different articles I have mentioned, along with many others, give us a glimpse of such a world, and ESM as a journal, has been instrumental in keeping such a vision alive.