

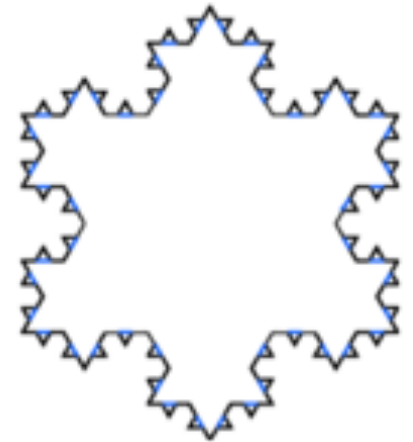
Adapted from MT 1810 Calculus II Unit from Rockhurst University Mathematics Department, Mairead Greene & Paula Shorter.

## Purpose:

To investigate the construction and area of a particular form of snowflake.

## Introduction:

In mathematics, we are often interested in describing the characteristics of shapes – whether that be perimeter (length), area or volume. In this task we will investigate some of these shape characteristics for a “mathematical snowflake”, defined using a recursive process (a process that repeats the same steps over and over again). A recursive process is often easy to carry out by hand for a few steps, but becomes very difficult for many steps. Recursive processes can create quite complex mathematical objects that we want to understand well. We study these recursively defined objects in two ways: (1) by using the computer to help us carry out many steps of the process, and (2) by noticing patterns in the process that we can take advantage of and use to study the object analytically.



## Part 1. Constructing Snowflakes – A Recursive Process.

We start our recursive construction process with an equilateral triangle. (Assume the triangle has sides of length 1 foot.) This triangle is our starting “snowflake”. Each stage in the recursive process is called an iteration. The triangle is iteration zero, which we label  $n=0$ . Carry out the next iteration (directly on the triangle) to create Snowflake  $n=1$  by altering each perimeter line segment of the triangle as follows:

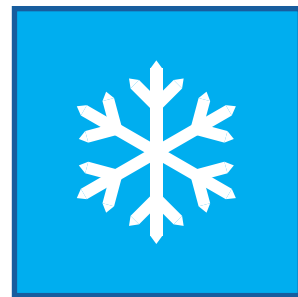
- a) Divide the line segment into three segments of equal length.
- b) Draw an equilateral triangle that has the middle segment from step (a) as its base and points outward.
- c) Remove the line segment that is the base of the triangle from step (b).

Once you have completed Steps (a) – (c) above for each perimeter line, you have created Snowflake  $n=1$ , the first iteration of the recursive process.

## Part 2. Shape Characteristics – Area:

(1) Now that we have constructed two snowflakes – the original triangle (Snowflake  $n=0$ ) and the first iteration (Snowflake  $n=1$ ) we will investigate the area of these snowflakes. Calculate the areas of these first two “snowflakes”.

(2) Repeat steps (a) – (c) in Part (1) for each perimeter line segment of Snowflake  $n=1$ . Once completed, you have created Snowflake  $n=2$ , the second iteration of the recursive process. Calculate the area of Snowflake  $n=2$ .



(3) We have now completed two stages (i.e., iterations) of the recursive process and you have created two new snowflakes in the process. We could use the same process to create many more snowflakes (look back at your last snowflake and think through how you would create the next snowflake). We are labeling each snowflake by its iteration number – i.e., how many times,  $n$ , we have performed the process (steps a-c in part 1). We are now interested in predicting the area of the next snowflake (Snowflake  $n=3$ ). The snowflakes have already become quite complicated, so rather than calculating this area by hand, think about the calculations you made in the first two iterations. Can you identify any patterns from your work in parts (2) and (3) above that will help us with this prediction? Write down all of the patterns that you find and be ready to share them with the class.

(4) Class Discussion: What patterns did you find that may be helpful in calculating the area of each new snowflake from its previous iteration?

(5) Predict the area that we will add on to the Snowflake  $n=2$  to get Snowflake  $n=3$ . What is the total area of Snowflake  $n=3$ ?

(6) Write a formula for the area that we add on at the  $n$ th iteration of the recursive process (when constructing Snowflake  $n$ ).

(7) Check your formula on the first three iterations of the recursive process. Then use your formula to help you calculate the total area of the fourth iteration (Snowflake  $n=4$ ).

(8) Express the total area of Snowflake  $n=4$  as the sum of 5 terms. (Hint: The first term should be the area of the original equilateral triangle.)

### Part 3. Exploring the Characteristics of the Koch Snowflake

This task allows further investigation of the characteristics of the Koch Snowflake with opportunity to consider finite sums and limits of finite sums (series), approximation of exact limit with finite sums, and some error considerations.



## Practice with Sigma Notation:

$$\sum_{k=1}^4 k = 1 + 2 + 3 + 4 = 10$$

$$\sum_{k=1}^5 2k^2 = 2 \cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 + 2 \cdot 4^2 + 2 \cdot 5^2 =$$

$$\sum_{k=1}^3 k(k+1) =$$

$$\sum_{k=0}^4 \frac{3^k}{2} =$$

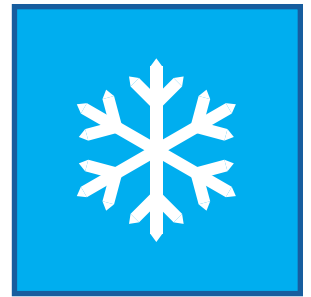
Write the following sums using sigma notation:

$$3 + 6 + 9 + 12 = \sum$$

$$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} = \sum$$

$$1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} = \sum$$

Desmos has a sigma summation in the keypad (bottom left corner) under the Functions button at the bottom of the Misc tab. Use this to check that you have written the sums above correctly in sigma notation.

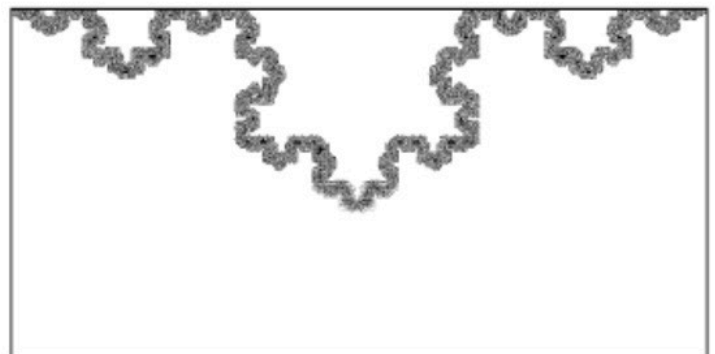
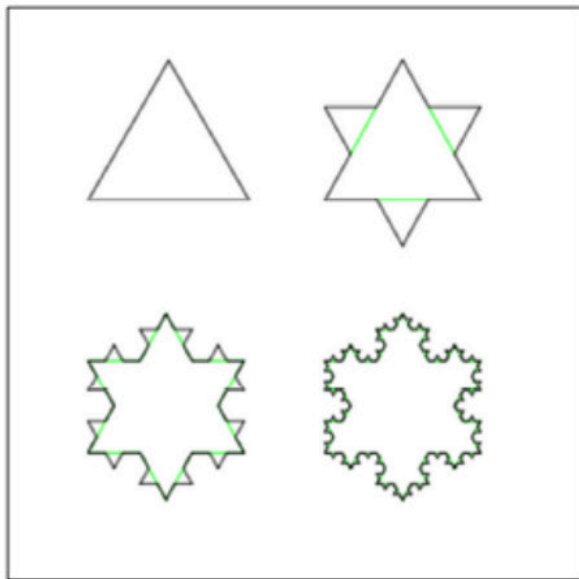


## The Koch Snowflake

The objects that you have been recursively constructing are called Koch Snowflakes, named after the Swedish mathematician who first studied them, Niels Fabian Helge von Koch (1870 – 1924). The starting triangle and the first three iterations of the snowflake are shown in the figure on the left below. These mathematical shapes are stages leading to the Koch curve, one of the earliest fractal curves to be described. If the recursive process that you went through in Part I of this activity was carried out infinitely, the resulting object would be the Koch curve. A section of the Koch curve is shown to the right below. When magnified, it has an infinitely repeating self-similarity. (This characteristic is a feature of all fractals.)

1. Write an expression for the area of Snowflake  $n = 4$  using sigma notation. Using Desmos, check this expression against the area that you have already found for Snowflake  $n = 4$  in the last activity.
2. Calculate the area of Snowflake  $n = 20$ .
3. Use the patterns that you recognized in Part I of this activity to write a formula for the area of the  $n^{\text{th}}$  Koch Snowflake using sigma notation:

$A(n) =$





## Class Discussion:

As described above, the Koch curve is the limit of an infinite recursive process - where in each iteration, additional area is added to the previous snowflake. Do you think the area enclosed by the Koch curve is finite or infinite? Explain your reasoning and be ready to share this with the class.

4. Desmos Investigation: With the help of Desmos, we will numerically investigate the question of whether the Koch curve encloses an infinite or finite amount of area. Now that you have developed a formula for the area of the  $n$ th Koch Snowflake, you can use a computer to easily calculate the area of any Koch snowflake.

Calculate several areas to investigate the question posed earlier: Is the area enclosed by the Koch curve finite or infinite? After you have completed your investigation in Desmos, state your conjecture below and answer the questions that follow.

a) Conjecture:

b) Explain how your results in Desmos support your conjecture:

c) If you have conjectured that the area enclosed by the Koch curve is finite, give an estimated value for the exact area enclosed by the Koch curve. What did you do to obtain this estimate?



5. Class Discussion: Exact Area Enclosed by the Koch Curve: You now realize that the area enclosed by the Koch curve is a series – the sum of an infinite number of terms. Every series is defined as the limiting value of the sequence of partial sums – here the partial sums are the increasing areas of the stages of the Koch Snowflake. Write the area enclosed by the Koch curve using a series in sigma notation:

Area enclosed by the Koch curve =  $\sum$

6. Approximations and Error:

Use Wolfram Alpha to calculate the exact value for the expression above representing the area enclosed by the Koch curve. (For an example of using Wolfram Alpha to compute an infinite sum see: <https://www.wolframalpha.com/input/?i=sum+%28%2F4%29%5Ej%2C+j%3D0..infinity&lk=3>)

1. In Q5 you have calculated partial sum approximations for the exact area enclosed by the Koch curve – these are the areas of iterations of the Koch Snowflake. Use the  $n = 5$ ,  $n = 10$  and  $n = 50$  Koch snowflakes as your partial sum approximations and for each approximation, calculate the associated error using Desmos. (Error = exact value – approximation)
2. Suppose we need a partial sum approximation for the exact area enclosed by the Koch curve that is within  $.00001 = 10^{-5}$  of the exact area. What is the smallest  $n$  for which the area of the  $n$ th Koch snowflake will give us a sufficiently accurate approximation? Explain your reasoning.

**Class Discussion: What Have We Learned in this Activity?**

**Extension Questions**

1. Write a formula for the area of the 30th Koch Snowflake ( $n = 30$ ) using sigma notation.
2. Calculate the area of the 30th Koch Snowflake ( $n = 30$ ). (You may use Desmos.)
3. If we use the area of the 30th Koch Snowflake as an approximation for the area enclosed by the Koch curve, what is the error associated with this approximation? (You may use Desmos.)