Prove It to Me!

Engage your learners through tasks proven to significantly promote reasoning and problem solving, which touch on many of the Mathematics Teaching Practices in *Principles to Actions: Ensuring Mathematical Success for All*. These tasks are discussed in this article, another installment in the series.

Jo Boaler

Reasoning and problem solving are both critical acts in mathematics learning. I have met many teachers who tell me they would love to highlight them more but feel they cannot because of the pressures of content coverage, pacing guides, and district tests. In this article, I describe a teaching intervention that highlighted problem solving and reasoning and resulted in changed mathematics pathways for students, whether they were at the low, middle, or high end of the achievement spectrum. Two summers ago, my Stanford youcubed team and I invited 83 middle school students to the Stanford campus to learn mathematics differently. I knew from many research studies that students would be helped if they were actively engaged in mathematics, solving creative and rich problems, and if they believed in their potential (Boaler 2016; 2015). I did not, however, expect the huge positive change the students made in their achievement, their beliefs about themselves, and their relationship with mathematics.

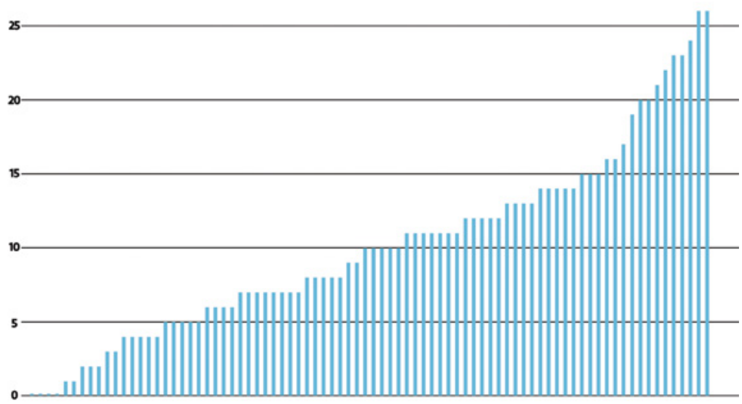
STRUCTURE OF THE YOUCUBED SUMMER CAMP

In preparation for the youcubed summer camp, I asked district math leaders if they could find students who had developed negative ideas about themselves and about math. They assured me that they could and sent the details of the camp to teachers

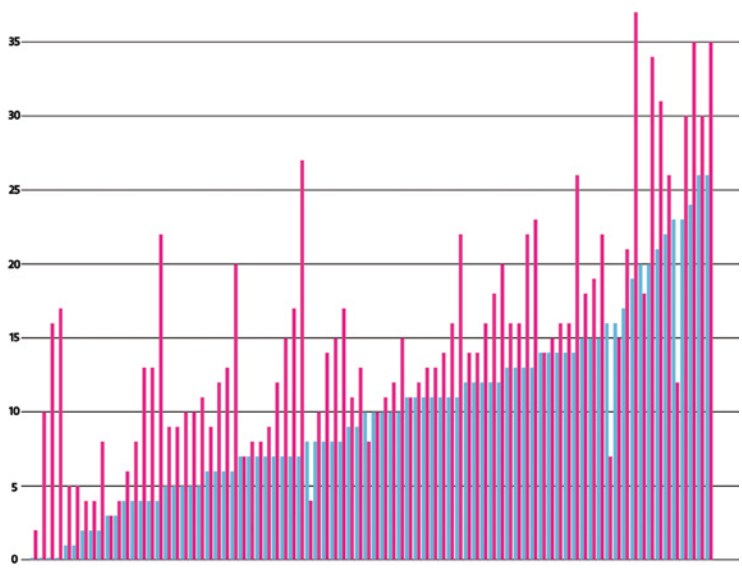
Table 1 Camp attendees would show improvement, regardless of gender and ethnicity.

Ethnicity and Gender	Girls	Boys
Asian	8	6
African American	1	0
Filipino	9	5
Latinx	10	9
Native Hawaiian/Pacific Islander	1	2
White	7	4
Mixed Race (2 or more)	9	12
Total	**45**	**38**

Fig. 1 The improvement of students in the youcubed summer school on standardized tests is shown in two graphs. The *y*-axis shows algebra scores; the *x*-axis shows each of the 83 students.



Pretest: An algebra test was taken before summer.

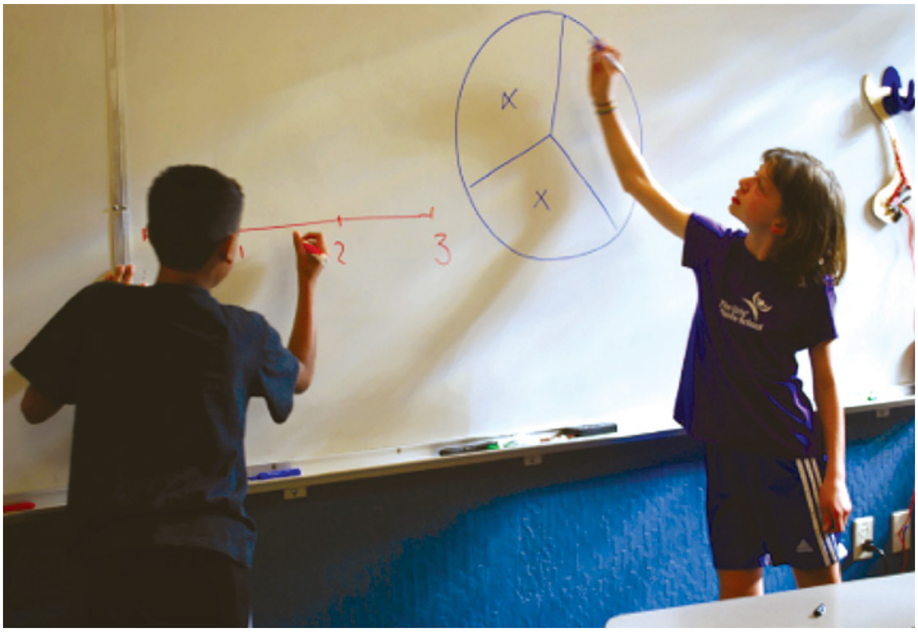


Posttest: The same test at the end of summer showed an average improvement of 50 percent.

of such students. The camp was free to students, held on the Stanford campus, and offered a free bus ride to and from the camp each day. In the mornings, students worked on math in four different classrooms; in the afternoons, they formed groups of about 20 students and spent time with Stanford undergraduates, seeing different parts of campus and working on such activities as scavenger hunts and taking photographs around campus. The 83 students who came to us that summer all told a researcher when they arrived that they were "not a math person." They could name the one person in their class who they thought *was* a "math person": the student who shot up a hand first when a question was asked. Students held this faulty belief even though they came from a wide range of achievement levels. **Table 1** shows the gender and ethnicity of the students. The myth that being fast in math is important was one of many that we dispelled for the students. We also shared with them the new and important brain science showing the following:

1. Nobody is born with or without a math brain, and brains can grow and change to learn any level of school mathematics.
2. Making mistakes and struggling are the best times for brain growth.
3. Thinking visually and making connections between different representations is important for brain connections and growth.

For the next eighteen lessons, we taught students using tasks that were open, creative, and promoted reasoning and problem solving. At the end of the eighteen lessons, the improvement of the students on standardized test scores was equivalent to 2.8 years of school. Despite the diversity of the pool of students who participated in the camp, a regression analysis shows that the improvement in performance did not differentiate by gender or by ethnic group (see **fig. 1**). You can watch a short film showing the approach at www.youcubed.org. The curriculum we used in our summer school is available to those who attend our workshops at Stanford, and our website (https://www.youcubed.org/tasks/) shares many of these tasks, freely available for anyone to use.



Students reason—with visuals—to solve 1 divided by 2/3.

THE YOUCUBED CAMP APPROACH

The curriculum of the youcubed camp was planned around four "Big Ideas" that we regarded as critical for the students' mathematics learning (see **fig. 2**). Many of the smaller ideas and methods arose naturally when we taught to the Big Ideas. (See also https://tinyurl.com/bigideaspaper, an article that shows big ideas for all K–8 mathematics.)

We wanted to teach students how to generalize and to use algebra as a problem-solving tool, and we introduced these ideas through visual pattern tasks. We encouraged students as they worked to make connections between the visual representations and numbers, thus encouraging brain connections. We also reminded students of the importance of developing brain connections by seeing mathematics in different ways: visually, numerically, algebraically, verbally, and in tabular or pictorial form. When students found the work difficult, we encouraged them with other brain science, telling them that times of struggle are the most important times for our brains.

To help students learn to reason mathematically when solving problems, we not only provided tasks that encouraged problem solving and reasoning but also taught them ways to reason with one another. We told students that reasoning is very important in mathematics and that it is an intrinsically mathematical act. Whereas scientists prove hypotheses by finding confirmatory or disconfirmatory evidence, mathematicians prove conjectures by reasoning—talking about why methods are chosen, how they work, and how they link to one another, describing the logical connections among them. I often meet parents of high-achieving students who ask me, "Why should my students explain their work when they can just get the answer?" My response is always the same: Explaining work is what we call in mathematics *reasoning*, and if students are not reasoning, they are not being mathematical. In conversations with Conrad Wolfram, leader of Wolfram-Alpha, the computational knowledge engine, he told me that he was only interested in employing people who can communicate mathematically and reason about their ideas. People who can only arrive at solutions are not productive in teams of people working on high-level mathematics.

Fig. 2 Many smaller ideas and methods arose naturally when we taught to the Big Ideas around which we planned the youcubed camp curriculum.

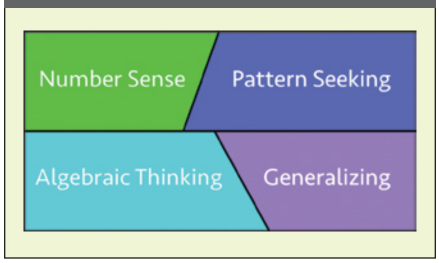
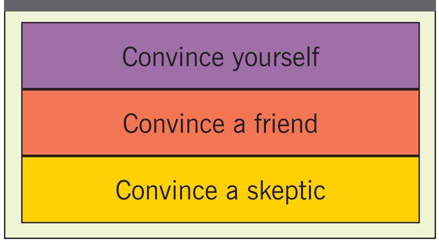


Fig. 3 In Humphreys's skeptics framework, the easiest level of persuasion is to convince oneself.



To encourage students to reason with one another, we taught them the skeptics framework that I learned from Cathy Humphreys. We told students that it was really good to be convincing and that the easiest level of convincing is to convince yourself of something, the next level is to convince a friend and the highest and hardest level is to convince a skeptic (see **fig. 3**).

We then asked students to *be* skeptics and suggested asking questions when others explained their thinking:

Fig. 4 Driscoll task instructions are listed.

For each part of the problem, start with a square sheet of paper and make folds to construct a new shape. Then explain how you know the shape you constructed has the specified area.

1. Construct a triangle with exactly 1/4 the area of the original square. Convince your partner that it has 1/4 of the area.
2. Construct another triangle, also with 1/4 the area, that is not congruent to the first one you constructed. Convince your partner that it has 1/4 of the area.
3. Construct a square with exactly 1/2 the area of the original square. Convince your partner that it is a square and has 1/2 of the area.
4. Construct another square, also with 1/2 the area, that is oriented differently from the one you constructed in number 3. Convince your partner that it has 1/2 of the area.

- How do you know that works?
- Why did you use that method?
- Can you prove that to us?

Students loved the role of the skeptic and readily took it on, and the teachers immediately saw positive changes in the teaching environment. Students challenged one another in a friendly and joking way to "prove it." Students then reasoned more completely. The two students below reflect on the skeptic role, joking with each other about mathematical proof:

Int.: So, what did it take in summer math camp to be successful?
TJ: Being able to communicate with your partner as you go.
José: And being able to show visuals, not just numbers.
TJ: Being able to explain things well.
José: And then someone says, "How" or "Why" or—
TJ and José: Prove it! [laughing].
José: Uh, what, what is that called, a, um—
TJ: Skeptical question.
José: Yeah, skep—, yeah, skeptic.
Int.: And what does that mean, and how does that feel?
TJ: It's fun to be.
José: [Laughs]
Int: Can you explain?
TJ: Because, like, it helps the other person that's not being skeptical . . .
José: Think about the problem.
TJ: Yes. For example, if Carlos said, like, "This is a square," and I'm, like, "Prove it."
José: Mmm, it has all, um, it, OK, it has all even sides and all, and all the corners are 90 degrees.
TJ: Why?
José: 'Cause it is.
TJ: Prove it!
José: It is! [Laughs]
TJ: [Laughs]
José: I just proved it.

At first, José jokes with TJ, saying " 'cause it is," but then he reminds TJ that he had already proved it with his mathematical statement ("It has all even sides and all, and all the corners are 90 degrees").

A task we used to teach problem solving and reasoning is one we have adapted (see **fig. 4**) from a Mark Driscoll task, and we share it on our website: https://www.youcubed.org/tasks/paper-folding/. Driscoll designed a lovely task that has been enjoyed by many of the students and teachers I work with, in which students are asked to fold shapes to show geometric properties in increasingly more difficult questions. We adapted Driscoll's task slightly by suggesting that students work in pairs and take turns being the paper folder/reasoner or the skeptic who asks questions and pushes the other student to reason to higher levels. Students enjoy taking turns as they move through the questions, to either be the one folding the paper and reasoning or the skeptic.

When we introduced the task to students, we explicitly modeled for them the sort of questions a skeptic may ask to encourage higher levels of reasoning. We discouraged them from merely accepting that their partner had proved the conditions and instead suggested asking questions, such as, "How do you know that is an equal triangle?"

Any mathematics task or question can be adapted to promote reasoning. Students can be asked to calculate 1 divided by 2/3, or they can be asked to show a visual proof of their solution to 1 divided by 2/3 (Boaler and Humphreys 2005). In the first version of the question, students are performing a calculation; in the second, they are drawing, which is an important part of mathematical thinking (see Boaler et al. 2016) and reasoning about their ideas.

REASONING AND EQUITY

From different research studies, I have found that reasoning plays an important role in the promotion of equity. This is partly because when students talk openly about the mathematical decisions they make, it helps others who are less sure, and it closes the gaps between the understanding of different students. In our summer camp, we also found that the open and visual nature of the tasks in which students were engaged increased opportunities for equitable group work. Teachers often worry about group work when students do not engage equally and some students do all the work while others do not or are excluded. This did not happen in our group work because students started each task by asking one another, "How do you see it?" They went around the group, finding out how different group members saw the mathematics. This helped all the students to feel included and invested in the group work that followed. I often shared with students that some of the most important and beautiful aspects of mathematics are the diversity in ways of seeing mathematics and in ways of solving problems. As teachers, we always valued different approaches, strategies, methods, and visuals, so students learned to value the different ways all their group members saw and solved tasks.

As students come to appreciate the different ways that people see and solve problems, rather than fast thinking or correct answers, they start to appreciate their peers, whatever their prior achievement. Elsewhere I have referred to this as "relational equity" (Boaler 2006). Students shifted their perspectives over the course of summer camp, changing their minds about their own potential and about the nature of mathematics. They started to see themselves as capable, and they saw their role in mathematics as problem solvers, people who explored conjectures and ideas, and reasoned about them.

DISCUSSION AND CONCLUSION

I opened this article by noting the significant change in students' mathematics achievement after attending the camp. The change in students' achievement came about, in part, because of a change in their confidence as they started to rethink who they were and what they were capable of doing. This quote from a student, Selina, interviewed at the end of the camp reflects the two ways in which students changed their perspectives:

> They taught us how, um, how math is for everybody; and I believed that I wasn't a math person before, but now I believe that anybody can do math, and, and that helped me a lot. And the way I thought that math was all about right answers and wrong; but it's really about ideas, and it's very creative, and that helped me like it a lot more.

Many teachers have embraced the growth mindset movement, sharing with their students the importance of knowing that they can learn and improve and that brains are not fixed. But, as both mindset guru Dweck (Gross-Loh 2017) and author Kohn have pointed out (2015), it is not enough to share these messages and then teach in the same way. If we tell students to try hard but do not show them ways of getting increased access or if we present mathematics as a set of short, closed questions, we run the risk of giving mixed messages. For students to see mathematics as a growth subject, they need mathematics questions through which they can grow and that give them many ways to be successful; that is, questions that ask students to reason, draw, collaborate, make connections, and learn. For example, 1 divided by 2/3 can be a fixed mindset question in which there is one answer and one way to approach it, or it can be a growth-mindset question when students are asked to reason, visualize, and discuss their ideas, thus giving opportunities for learning and growth (see also Boaler 2016).

Before coming to summer camp, the students had worked in silent mathematics classrooms, passively "receiving" knowledge. When they were invited, in summer camp, to reason and make sense of mathematics, it was liberating for

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Reasoning and Problem-Solving Resources

The following list of useful websites provides some additional problem-solving possibilities.

- www.youcubed.org
- http://www.visualpatterns.org/
- http://blog.mrmeyer.com/
- http://www.nctm.org/
- https://turnonccmath.net/
- http://map.mathshell.org/
- https://www.illustrativemathematics.org/
- http://www.estimation180.com/
- http://mathpickle.com/

> Although we did not prepare students for the content of the test, they worked to make sense of different questions and reason their way through them, believing that they could succeed.

many. A key part of the transition for students came from the teachers making mathematics, rather than themselves, the authority in the classrooms. When questions were asked, the teachers did not answer them; instead they asked students to reason about them, drawing on their mathematical thinking. In doing so, the classroom became a community of learners, with all participants, including the teachers, taking a role in the learning, as equals. As the eighteen lessons progressed, the students changed their ideas about who they were as people and about their role in mathematics learning. These changes are reflected in the students' achievement during the eighteen days and their improved test scores. Although we did not prepare students for the content of the test, when they took the test, they worked to make sense of different questions and reason their way through them, believing that they could succeed.

The change that took place for students during the summer is one that is available to all students and teachers, particularly when teachers have opportunities to learn about the new brain science and have access to open, creative mathematics tasks that invite students to reason and problem solve. Our youcubed website offers free access to both, and I invite all readers to visit this and other important sites (some of which are listed in the **sidebar** on the previous page). When we make these changes for students, it not only improves students' test scores but also changes who they are as people. If we want our students to become the young adults our society needs—those who engage in twenty-first-century thinking by reasoning, connecting, and collaborating—then mathematics classrooms must become places in which students believe in their own unlimited potential and engage actively with mathematical ideas. Such classrooms are more exciting for students and for teachers. We are all mathematics learners, and we can all develop active, inquiring relationships with mathematics. When we do, and mathematics becomes a creative, open space of inquiry, mathematics learners will find that they can do anything, and their mathematical ideas and thinking can extend to the sky—and beyond!

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