Visual Mathematics Activities

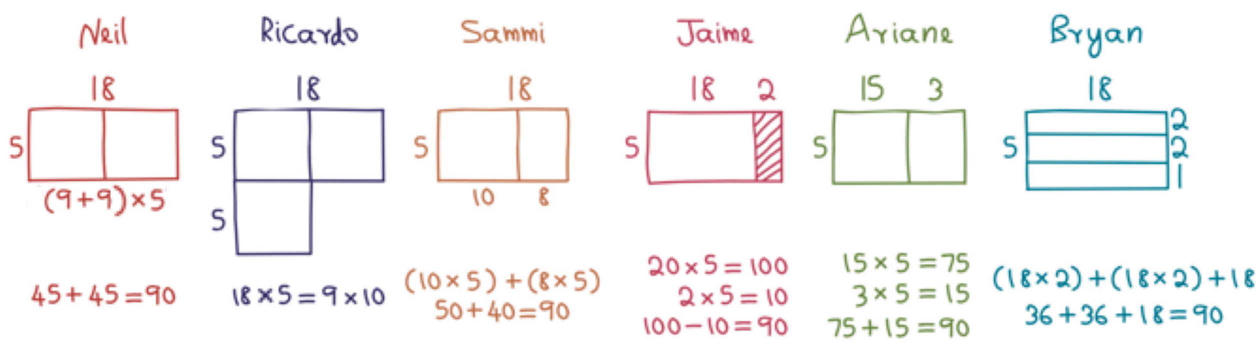
These activites go with the paper:

SEEING AS UNDERSTANDING: The Importance of Visual Mathematics for our Brain and Learning.

Jo Boaler, Professor of Mathematics Education
with Lang Chen, Stanford Cognitive and Systems Neuroscience Lab
Cathy Williams & Montserrat Cordero, youcubed.
Stanford University

http://www.youcubed.org/visual-math-network/

Work out 18 x 5 and show a visual solution.



from Jo Boaler. Mathematical Mindsets (2016)

How Close to 100?

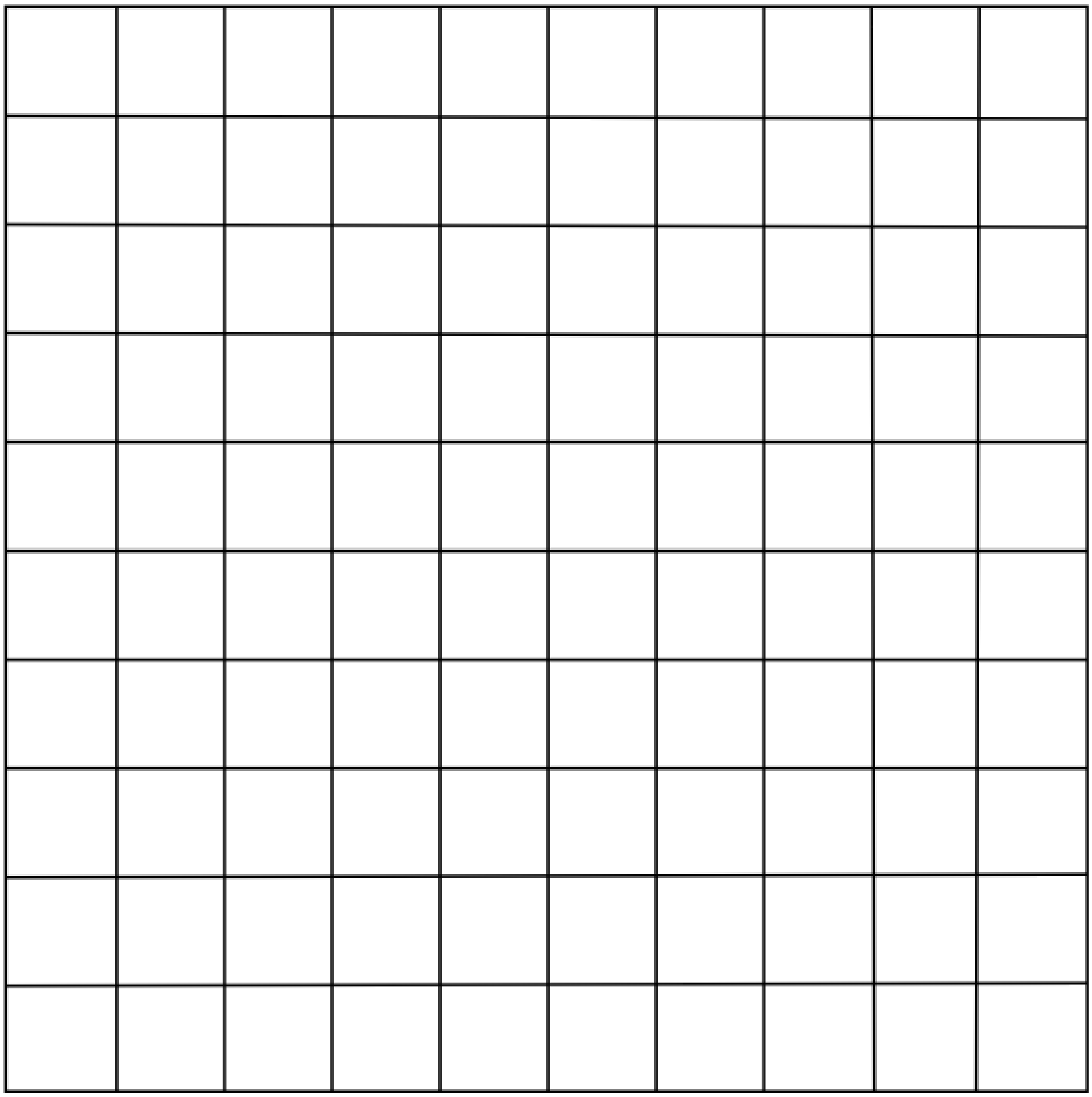
You will need:
- two players
- two dice
- recording sheet (see next page)

This game is played in partners. Two children share a blank 100 grid. The first partner rolls two number dice. The numbers that come up are the numbers the child uses to make an array on the 100 grid. They can put the array anywhere on the grid, but the goal is to fill up the grid to get it as full as possible. After the player draws the array on the grid, she writes in the number sentence that describes the grid. The second player then rolls the dice, draws the number grid and records their number sentence. The game ends when both players have rolled the dice and cannot put any more arrays on the grid. How close to 100 can you get?

Variation:
Each child can have their own number grid. Play moves forward to see who can get closest to 100.

from Jo Boaler. Fluency Without Fear.

How Close to 100?



1. _______ x _______ = _______
2. _______ x _______ = _______
3. _______ x _______ = _______
4. _______ x _______ = _______
5. _______ x _______ = _______
6. _______ x _______ = _______
7. _______ x _______ = _______
8. _______ x _______ = _______
9. _______ x _______ = _______
10. _______ x _______ = _______

The Border Problem

You will need:

Border problem image to display to class (attached)

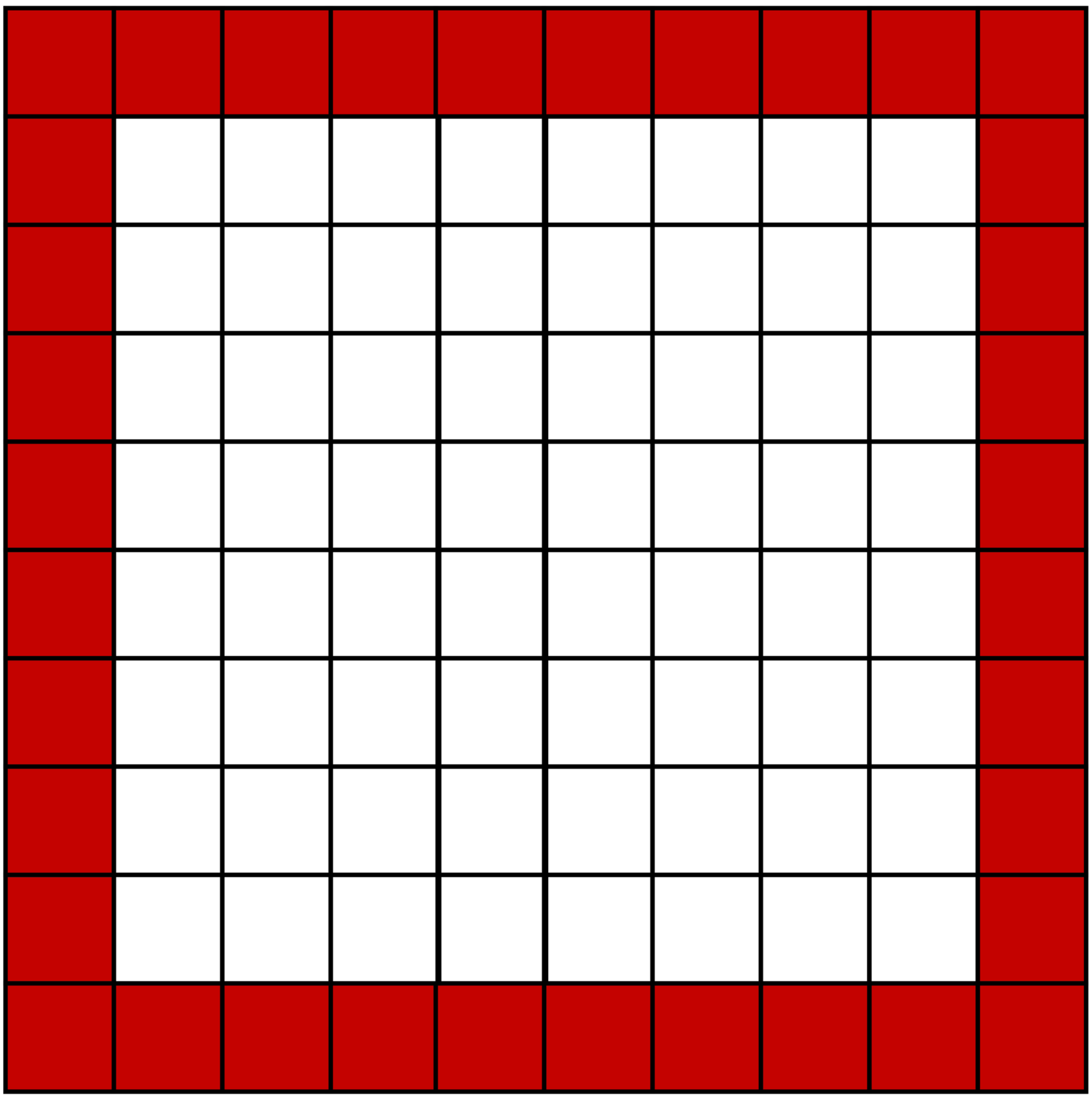
1. Display the border problem image to the whole class very briefly. Tell them it is a 10 x 10 grid and ask them to work out the number of squares in the border without conting one by one. Don't give the students their own copy of the border. Do not leave it up long enough for them to count.
2. Ask students to share their answer with a partner without discussing how they got their answer.
3. Ask students to share their methods with the entire class. Record their responses using drawings to illustrate their thinking (see
4. When all methods are displayed ask students to compare and contrast the different methods.

Extensions:

1. Ask students to represent algebraically
2. Discuss the equivalence of the algebraic expressions
3. Ask students to shrink, or extend the grid and to think about the borders of different sized grids, leading to algebraic expressions.

For more details about how to facilitate this problem and a video of Cathy Humphrey teaching it see: Boaler, J., & Humphreys, C. (2005). Connecting mathematical ideas: Middle school video cases to support teaching and learning (Vol. 1). Heinemann Educational Books.

The Border Problem



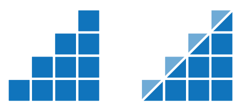
youcubed at Stanford University

algebra

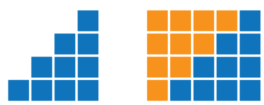
Squares and Cubes

Visuals for Deep Understanding

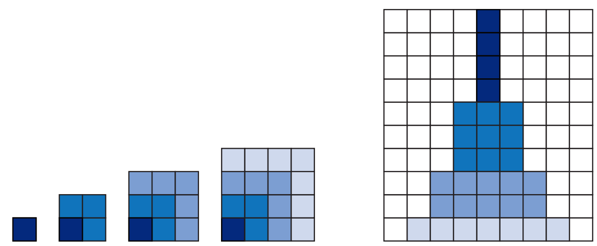
How do the diagrams explain the equations?
How do the diagrams and equations change as more terms or fewer terms are included in each series?



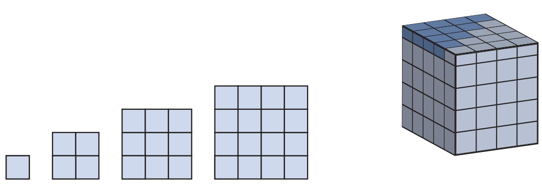
$$1+2+3+4 = \frac{4^2}{2} + \frac{4}{2}$$



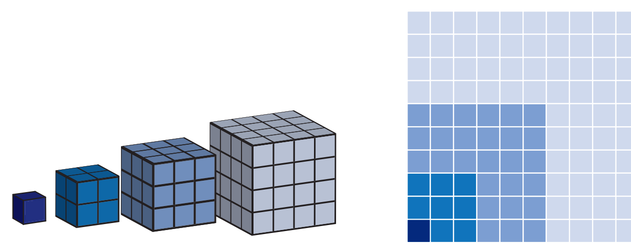
$$1+2+3+4 = \frac{4 \times (4+1)}{2}$$



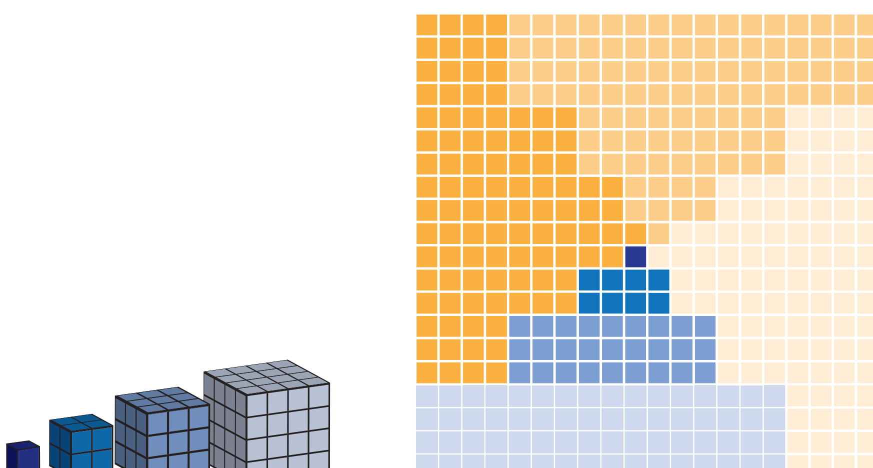
$$1^2 + 2^2 + 3^2 + 4^2 = \frac{(2\times 4 + 1) \times (1 + 2 + 3 + 4)}{3}$$



$$1^2 + 2^2 + 3^2 + 4^2 = \frac{4\times(4+1)\times(4+1/2)}{3}$$



$$1^3 + 2^3 + 3^3 + 4^3 = (1+2+3+4)^2$$



$$1^3 + 2^3 + 3^3 + 4^3 = \frac{4\times(4+1)^2}{4}$$

Problem provided by Gary Antonick, http://wordplay.blogs.nytimes.com/tag/visual-thinking/

linear transformations

Draw the parallelogram spanned by the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Then draw its image under the linear transformation defined by the following matrix: $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$.

How are these different from one another?

What do they still share in common?

How could you change the matrix A to change each of those characteristics in the image?

youcubed at Stanford University

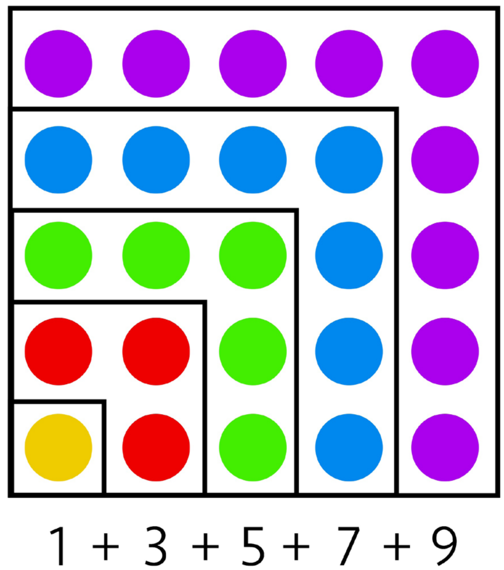
Surprising Visual Solutions and Proofs
number

Odd Numbers

The sum of consecutive odd numbers beginning with 1 has a unique and wonderful pattern!

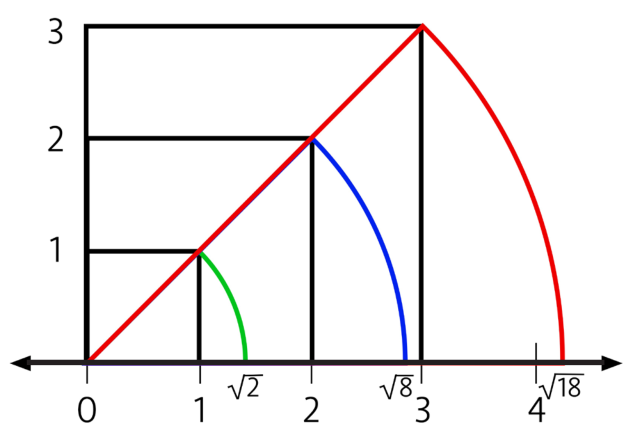
1 + 3 = 4
1 + 3 + 5 = 9
1 + 3 + 5 + 7 = 16
1 + 3 + 5 + 7 + 9 = 25

Does the pattern continue?



Irrational Numbers

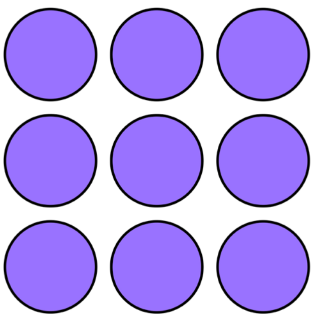
Squares, their diagonals and a numberline make irrational numbers really interesting!

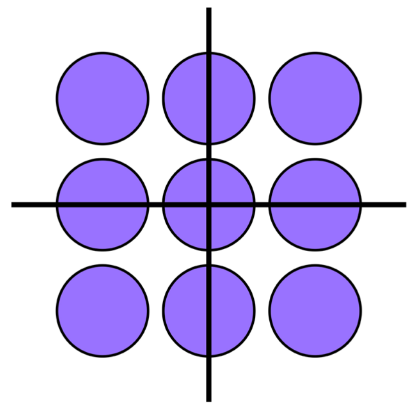


Surprising Visual Solutions and Proofs
algebra

The Turkey Problem

A man is on a diet and goes into a shop to buy some turkey slices. He is given 3 slices which together weigh ⅓ of a pound but his diet says that he is allowed to eat only ¼ of a pound. How much of the 3 slices he bought can he eat while staying true to his diet?



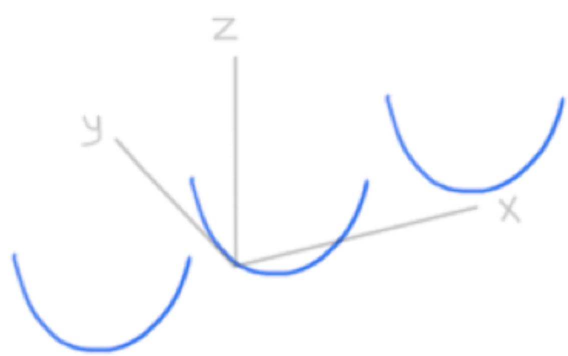


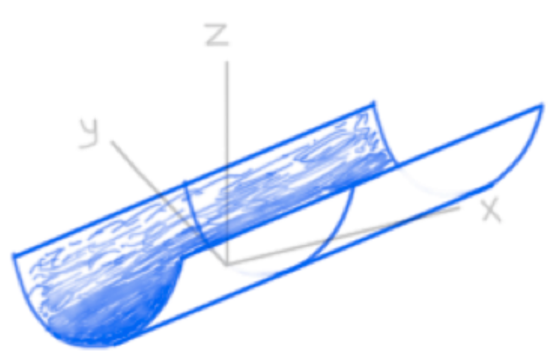
from Jo Boaler. Mathematical Mindsets (2016)

3-D Parabolas

Sketch the three-dimensional surface and level curves of $z = y^2 + x$.

Solution:
Because z's dependence on x is simpler than its dependence on y, we plot cross-sections with three values: $x = 0$, x negative, and x positive.





Notice that the minimal value of each parabola is $z = x$. For visual clarity, the figure is not drawn to scale.
We observe that the survace is like a parabola in (y,z) that slopes along the x axis. This shape is similar to a half-pipe on a ski slope.

youcubed at Stanford University

Surprising Visual Solutions and Proofs

probability

Uber/Lyft

by Gary Antonick

Math is the science of patterns and the language of logic. We use math to track quantities and trends, like the money in our savings account. We use math to create: engineers and chefs rely on plans that specify exact quantities. And we use math to help make decisions -- to provide a way to check our initial intuition.

Recent studies in psychology have shown that our initial intuition is often wildly incorrect. Try this example:

> Uber and Lyft
>
> In your city there are two ride-share companies: Uber and Lyft. Your father uses one of these to get back from the airport, but leaves his phone in the car! You are given the following data:
>
> • 85% of the cabs in the city are Uber and 15% are Lyft.
>
> • Your dad thinks he left his phone in a Lyft car, but he's not sure. In your experience your dad is correct about 80% of the time, and incorrect about 20% of the time.
>
> He wants to get his phone back. Which company should he call first? That is, is it more likely that he left his phone in an Uber car or a Lyft car?

This problem is a classic in conditional probability, which is an especially useful tool in fields such as law and medicine. Why is the mathematics of conditional probability so important? "Human beings have not evolved to solve these kinds of problems," says Nick McKweown, a professor of computer science at Stanford. "We need the rigor of math to save us from ourselves." Let's see why.

Our initial intuition

Our initial intuition is — call Lyft. Your dad thinks he left his phone in a Lyft car, and his memory is pretty good for things like this: 80% accurate.

That's our initial intuition, but it's actually incorrect. The phone is actually more likely to be in an Uber car! But how could this be?

Solving with a formula

By using a formula for conditional probability, we find the chance of phone being in a Lyft car is actually only 41%. Therefore it's more likely to be in an Uber car. Math saved us! Our intuition is corrected.

The problem, though, is that it's hard to believe the 41% is actually correct. Your dad's right 80% of the time. How does this even make sense? Did we make a mistake somewhere? This is is where visual math can help.

Uber/Lyft

Solving visually using concrete shapes

One way to visualize this is to use actual little cars.

Here they are: 100 cars. 85 are Uber, 15 are Lyft.

Your dad is 80% reliable. Let's forget about Uber and Lyft for a minute. How do we draw "80% reliable?" One way — let's draw 80 dark gray cars and 20 light gray cars. One of these cars has the phone inside, but we don't know which. If your dad guesses "dark gray," he'll be correct 80% of the time if we repeated this and put the phone in a different car each time. This time we shade the cars by row.

Now we can combine these two ideas. A dark color means your dad chose correctly, and a light color means he chose incorrectly.

Let's now solve our problem. If your dad thinks he left the phone in a Lyft car, what's the probability the phone is actually in a Lyft car? In this case, the situation is either Lyft/Correct or Uber/Incorrect. The chance the cab is Lyft/Correct is the proportion of correct Lyft cars to total cars in the diagram, or 12/(12+17), or 12/29. About 41%. So even though he thinks he left the phone in a Lyft car, the phone is more likely to be in an Uber car. He should call Uber first.

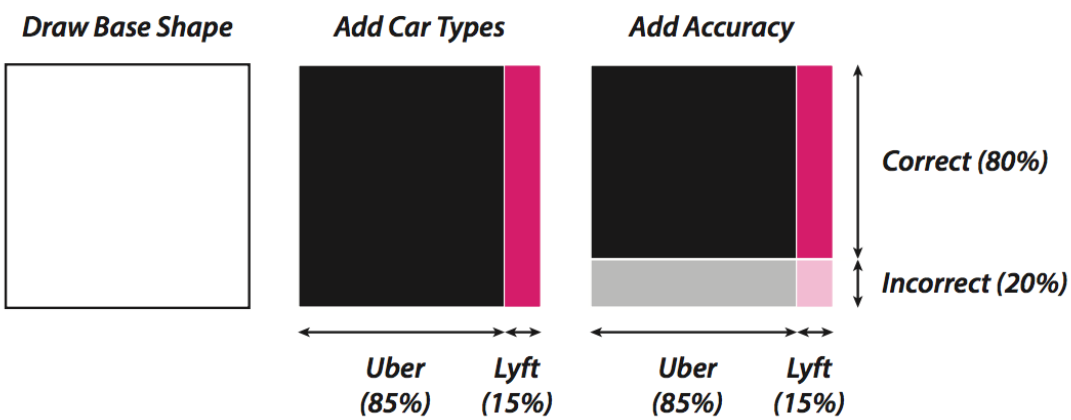
We can now see what's going on. Your dad is 80% accurate, but that's his overall average, which includes when he thinks the car is an Uber. His accuracy in this case (when he thinks the car was an Uber) must be way higher, and we can confirm this is so: 68/(68+3) = 96%.

We now have the opportunity to understand even more about this problem in ways that would be very difficult with a formula. What happens, for example, if there are three different ride-share options in your city? How does this change the problem?

Uber/Lyft

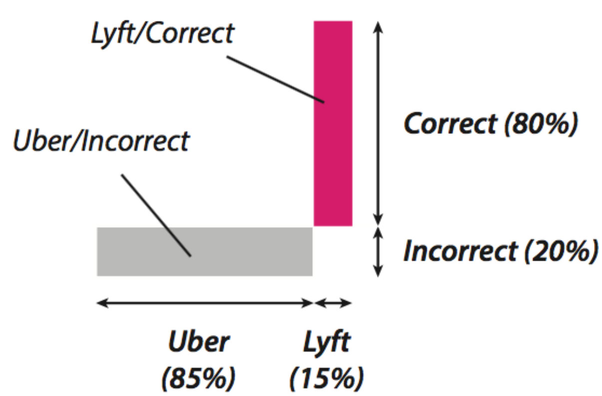
Solving visually using rectangles

A second way to visualize this Uber/Lyft problem is by using rectangles. The basic idea is the same as with the icon solution on the previous page, but adds an additional level of abstraction.



We can now solve our problem. If your dad thinks he left the phone in a Lyft car, what's the probability the phone is actually in a Lyft car?

Your dad said Lyft and is either correct or incorrect, of course. The relevant regions of this diagram are Lyft/Correct and Uber/Incorrect. The probability of being Lyft/Correct is the area of the pink rectangle divided by the area of both triangles.



Our calculation is

pink/(pink + gray) =
(15×80)/(15×80 + 85×20) =
1200/(1200 + 1700) =
1200/2900 =
0.4138

And we're done! The probability that your dad left his phone in a Lyft car is 41%. So he should call Uber first.

Problem provided by Gary Antonick, http://wordplay.blogs.nytimes.com/tag/visual-thinking/